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Boundary Excursions for Combined Random Loads

J. R. Fuller*
The Boeing Company, Seattle, Wash.

Components of internal loads in a structure, such as shear, moment, and torsion, resulting from externally applied random loading are also random and are generally statistically correlated. The strength of a structural section is related to the magnitude of the internal load components through an interaction function. The procedure derived herein allows the determination of the number of excursions through the interaction strength boundary per unit time.

Nomenclature

$\bar{A}_x, \bar{A}_y, \bar{A}_z, \ldots$	= rms values of $x,y,z,\ldots$ for unit σ_w
$\bar{A}_\alpha, \bar{A}_\beta, \bar{A}_\gamma, \ldots$	= rms velocities of $x,y,z,\ldots$ for unit σ_w
C	= limit design load function
$\nabla C, \nabla S$	= gradient vectors on C and S
$F_x, F_y, F_z, \ldots$	= allowable limit design loads
$I_{(+x)}, I_{(-x)}, \ldots$	= integral of probability density evaluated on the surface C or S and integrated over the projection of C or S on a plane normal to the $+x$ axis, $-x$ axis,...
$[\bar{\ell}(t), \bar{v}(t)$	= random load and load velocity vectors
$\bar{N}$	= unit normal vector
$N_c(\hat{x}), N_c(\hat{y}), \ldots$	= number of excursions per unit time with positive slope through $x,y,\ldots$
$N_c(\hat{x},\hat{y})$	= number of excursions per unit time out of boundary function of x and y
$N_c(\hat{x},\hat{y},\hat{z})$	= number of excursions per unit time out of boundary function of x, y, and z
$N_{0x}, N_{0y}, N_{0z}, \ldots$	= number of excursions per unit time with positive slope through x_0, y_0, $z_0,\ldots$
$[R]$	= correlation coefficient matrix
S	= limit design strength function
U_σ	= limit design rms excitation, $\eta_d \sigma_w$
$x,y,z,\ldots$	= components of internal loads
$x_0, y_0, z_0, \ldots$	= steady-state or mean values of x, y, $z,\ldots$
$\hat{x}, \hat{y}, \hat{z}, \ldots$	= time-varying part of x, y, $z,\ldots$
$\alpha, \beta, \gamma, \ldots$	= velocities of x, y, $z,\ldots$
$\rho_{xy}, \rho_{xz}, \ldots$	= statistical correlation coefficients between load components x and y, x and z, etc.
σ_w	= rms excitation
$\sigma_x, \sigma_y, \sigma_z, \ldots$	= rms values of x, y, $z,\ldots$
$\lfloor \; \rfloor$	= row matrix
$[\;]$	= column matrix
$\lceil \; \rfloor$	= diagonal matrix

Introduction

STRUCTURES are often subjected to random excitation such as atmospheric turbulence, turbulent boundary layers, wake turbulence, or other separated flow phenomena. These external loadings can set up severe random vibrations in all or part of the structure. In recent years, analytical tools have been developed to allow the designer to estimate the magnitudes and frequency content of the dynamic responses; also, random load design criteria have evolved that influence new designs.

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*Unit Chief, 747 Loads and Dynamics. Associate Fellow AIAA.

Generally, the designer is faced with scaling up a statistically described internal load level, such as an rms level of bending moment on a structural section, to obtain a limit design level. However, other components of loading on the section, such as shear and torsion, are also excited and affect the allowable bending moment. Obviously, the structure must sustain the combined random loads. A procedure is developed herein for defining the most critical combined random loads to ensure maximum structural efficiency.

Simply scaling rms levels up to limit design levels provides no insight into the number of exceedances of the limit design load envelope or the limit strength envelope that can be expected per unit time. Obviously, the number of exceedances of limit strength should be acceptably small within the projected useful life of the structure. A procedure is developed to compute the expected number of excursions through the limit design load envelope and the limit strength envelope or boundary per unit time. An example problem is shown to illustrate the calculation procedure.

Discussion

Design Envelope Approach for Combined Random Loads

Consider a linear elastic system subjected to stationary Gaussian random excitation. A limit design load in such a system can be established as a factor U_σ times the rms load resulting from an rms excitation σ_w of unity. Thus, the frequency density of the random loading can be depicted as

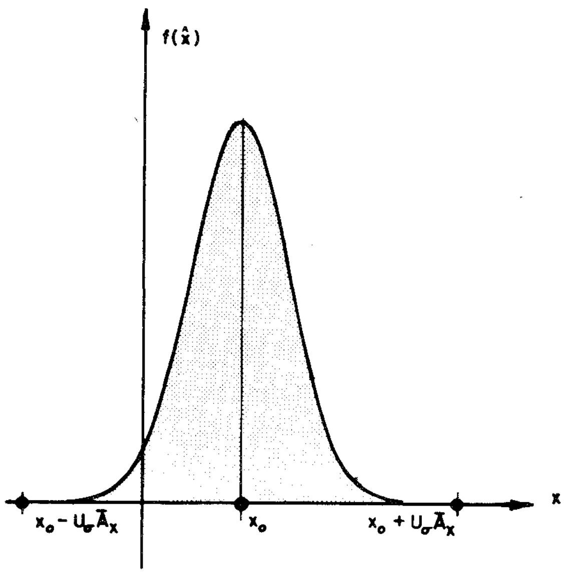

Fig. 1 Limit design loads for random loading $x(t)$.

shown in Fig. 1, where x_0 is the mean or steady-state component of the load x. The limit design loads are $x_0 \pm U_\sigma \bar{A}_x$, where $\bar{A}_x$ is the rms load due to unit σ_w.

When two load components are important in evaluating the strength of the system, a limit design load ellipse can be determined. This ellipse is a locus of equal frequency density and is shown in Fig. 2. The system must be able to sustain any combination of loads represented by points on the ellipse.

The two-dimensional Gaussian frequency density function is

$$f(\hat{x},\hat{y}) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho_{xy}^2)^{1/2}} \times \exp\left\{\frac{-1}{2(1-\rho_{xy}^2)}\left[\frac{\hat{x}^2}{\sigma_x^2} - \frac{2\rho_{xy}\hat{x}\hat{y}}{\sigma_x\sigma_y} + \frac{\hat{y}^2}{\sigma_y^2}\right]\right\} \tag{1}$$

It will be noted that the integral of $f(\hat{x},\hat{y})$ with respect to y is $f(\hat{x})$, shown in Fig. 1. Similarly, the integral of $f(\hat{x},\hat{y})$ with respect to x is $f(\hat{y})$.

The limit design load ellipse is inscribed between $x_0 \pm U_\sigma \bar{A}_x$ and $y_0 \pm U_\sigma \bar{A}_y$, where σ_x and σ_y are taken as $\bar{A}_x$ and $\bar{A}_y$, respectively,

$$U_\sigma^2 = \frac{1}{(1-\rho_{xy}^2)}\left[\frac{\hat{x}^2}{\bar{A}_x^2} - \frac{2\rho_{xy}\hat{x}\hat{y}}{\bar{A}_x\bar{A}_y} + \frac{\hat{y}^2}{\bar{A}_y^2}\right] \tag{2}$$

The Gaussian frequency density function for n load components[1] is

$$f(\hat{x},\hat{y},\hat{z},...,\hat{n}) = \frac{1}{(2\pi)^{n/2}\sigma_x\sigma_y\sigma_z,...,\sigma_n|R|^{1/2}} e^{-G/2} \tag{3}$$

where

$$G = \begin{bmatrix} \hat{x},\hat{y},\hat{z},...,\hat{n} \end{bmatrix} \begin{bmatrix} \ddots 1/\sigma_i^2 \ddots \end{bmatrix} \begin{bmatrix} R \end{bmatrix}^{-1} \begin{bmatrix} \ddots 1/\sigma_i^2 \ddots \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ \ddots \\ \hat{n} \end{bmatrix} \tag{4}$$

and the correlation matrix $[R]$ is

$$[R] = \begin{bmatrix} 1 & \rho_{xy} & \rho_{xz} & \cdots & \rho_{xn} \\ \rho_{xy} & 1 & \rho_{yz} & & \cdot \\ \rho_{xz} & \rho_{yz} & 1 & & \cdot \\ & & & & \cdot \\ \rho_{xn} & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{5}$$

Here again, the limit design loads are described by Eq. (4), where G is U_σ^2 and σ_i are set equal to $\bar{A}_i$. The approach described above is the so-called design envelope approach[2-4] for determining limit design load levels.

The whole procedure implied in the design envelope approach recognizes that there is a finite probability of exceeding the limit design loads. It is necessary to determine that the frequency of exceedance is acceptably low. The frequency of exceedance of the limit design load envelope is directly related to the so-called *characteristic frequencies* of the load components.

Frequency of Exceedance of the Limit Design Load Ellipse

Rice[5] showed that the number of crossings of a level x for a stationary Gaussian random time history can be related to the

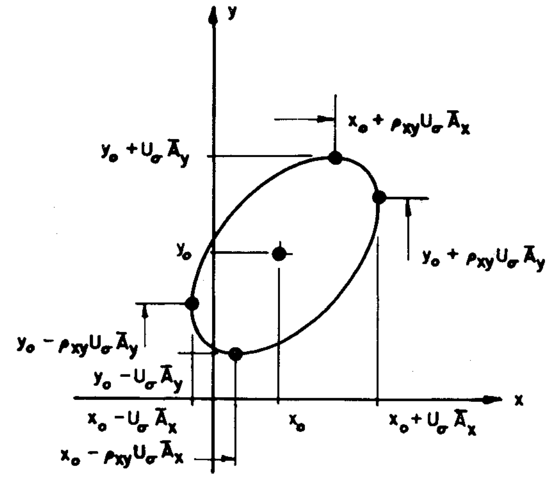

Fig. 2 Limit design load ellipse for random loadings $x(t)$ and $y(t)$.

rms level of x, that is, σ_x, and the rms velocity of x, that is, α or σ_α.

The joint probability density function for $\hat{x}$ and α, $f(\hat{x},\alpha)$, defines the percentage of time that the random time histories $\hat{x}$ and α are in an interval $(\hat{x}+\mathrm{d}x, \alpha+\mathrm{d}\alpha)$. The two time histories will be in the interval for the fraction τ of a unit time,

$$\tau = f(\hat{x},\alpha)\,\mathrm{d}x\,\mathrm{d}\alpha \tag{6}$$

The time required to cross the interval $\mathrm{d}x$ is $\mathrm{d}x/\alpha$; therefore, the number of crossings of $\mathrm{d}x$ per unit time with velocity α is

$$N_c(\hat{x})_\alpha = \frac{f(\hat{x},\alpha)\,\mathrm{d}x\,\mathrm{d}\alpha}{\mathrm{d}x/\alpha} = \alpha f(\hat{x},\alpha)\,\mathrm{d}\alpha \tag{7}$$

If only positive values of α, that is, positive slopes of the time history of x are considered, then the number of times per unit time that the time history of x passes through the interval $(\hat{x}+\mathrm{d}x)$ for all possible positive values of α is

$$N_c(\hat{x}) = \int_0^\infty \alpha f(\hat{x},\alpha)\,\mathrm{d}\alpha \tag{8}$$

If $\hat{x}$ and α are statistically independent, that is, the statistical correlation between $\hat{x}$ and α is zero, then Eq. (8) can be integrated readily to obtain Rice's equation,

$$N_c(\hat{x}) = \int_0^\infty \left(\frac{1}{\sqrt{2\pi}\sigma_\alpha}\alpha\exp\left[\frac{-\alpha^2}{2\sigma_\alpha^2}\right]\right)\left(\frac{1}{\sqrt{2\pi}\sigma_x}\exp\left[\frac{-\hat{x}^2}{2\sigma_x^2}\right]\right)\mathrm{d}\alpha$$

$$= \left\{\exp\left[\frac{-\hat{x}^2}{2\sigma_x^2}\right]\Big/2\pi\sigma_x\sigma_\alpha\right\}\int_0^\infty \alpha\exp\left[\frac{-\alpha^2}{2\sigma_\alpha^2}\right]\mathrm{d}\alpha$$

$$= \frac{1}{2\pi}\frac{\sigma_\alpha}{\sigma_x}\exp\left[\frac{-\hat{x}^2}{2\sigma_x^2}\right] \tag{9}$$

The statistical independence of $\hat{x}$ and its time derivative $\dot{x}$ or α can be proven by integrating (by parts) the time-averaged product of $\hat{x}$ and $\dot{x}$, which is an expression of the correlation between $\hat{x}$ and α,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \hat{x}(t)\dot{x}(t)\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\left\{[\hat{x}^2(t)]_0^T - \int_0^T \hat{x}(t)\dot{x}(t)\,\mathrm{d}t\right\} = \lim_{T\to\infty}\frac{1}{2T}[\hat{x}^2(T) - \hat{x}^2(0)] = 0 \tag{10}$$

This limit is zero, since the respective values of $\hat{x}^2(T)$ and $\hat{x}^2(0)$ are always finite, and they are divided by T, which is arbitrarily large.

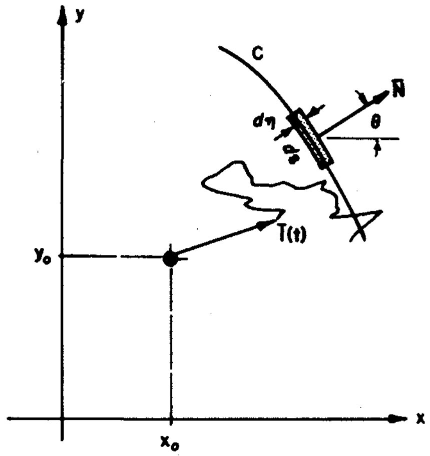

Fig. 3 Combined random load vector.

When x is equal to x_0 in Eq. (9), $N_c(\hat{x})$ is N_{0x}, the *characteristic frequency* of the random time history $x(t)$,

$$N_{0x} = (1/2\pi)(\sigma_\alpha/\sigma_x) \tag{11}$$

N_{0x} is the number of positive (or negative) slope crossings per unit time of $x = x_0$, the mean or steady-state value of x. There must be one maximum peak value of $x(t)$ for each positive (or negative) slope crossing.

The excursions per unit time through the limit design level of x are obtained from Eq. (11),

$$N_c(x_L) = (1/2\pi)(U_\sigma \bar{A}_\alpha / U_\sigma \bar{A}_x) e^{-U_\sigma^2/2} = N_{0x} e^{-U_\sigma^2/2} \tag{12}$$

Of course, the total number of excursions per unit time through limit design levels of x and $-x$ is twice that indicated in Eq. (12).

The number of excursions of a boundary in two dimensions can be derived in a manner similar to that developed by Rice[5] for a single random load component. The number of excursions per unit time of a boundary C in the xy plane[6-8] for a random load vector with components $\hat{x}(t)$ and $\hat{y}(t)$ is developed to show the excursions per unit time out of the limit design load ellipse, Eq. (2).

Consider the combined random load vector $\bar{\ell}(t)$,

$$\bar{\ell}(t) = \hat{x}(t) i + \hat{y}(t) j \tag{13}$$

and the corresponding velocity vector $\bar{v}(t)$,

$$\bar{v}(t) = \alpha(t) i + \beta(t) j \tag{14}$$

The fraction of unit time that the vector $\bar{\ell}(t)$ is in the interval $\mathrm{d}s\mathrm{d}\eta$ shown in Fig. 3 is

$$\tau = f(\hat{x},\hat{y},\alpha,\beta)\,\mathrm{d}s\,\mathrm{d}\eta\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{15}$$

The time required to cross the interval $\mathrm{d}\eta$ is equal to $\mathrm{d}\eta$ divided by the velocity component normal to C. The velocity normal to C is the scalar product of $\bar{v}(t)$ and the unit normal vector on C. The unit normal vector is defined, as shown in Fig. 3, as

$$\bar{N} = \cos\theta i + \sin\theta j \tag{16}$$

The velocity normal to C is the scalar product of $\bar{v}$ and $\bar{N}$,

$$\bar{v}\cdot\bar{N} = \alpha\cos\theta + \beta\sin\theta \tag{17}$$

The time required to cross the interval $\mathrm{d}\eta$ is $\mathrm{d}\eta/(\alpha\cos\theta + \beta\sin\theta)$; therefore, the number of crossings of C per unit time with velocity components α and β is

$$N_c(\hat{x},\hat{y})_{\alpha\beta} = \int_c \frac{f(\hat{x},\hat{y},\alpha,\beta)\,\mathrm{d}s\,\mathrm{d}\eta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\mathrm{d}\eta/(\alpha\cos\theta+\beta\sin\theta)} = \int_c (\alpha\cos\theta+\beta\sin\theta) f(\hat{x},\hat{y},\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}s \tag{18}$$

The sense of the unit normal vector on C is taken toward the exterior of the boundary. Therefore, positive values of α imply outward or positive velocities in the first and fourth quadrants, and negative values of α imply outward velocities in the second and third quadrants. The outward velocity component normal to C can be expressed by positive α and β and the absolute values of $\cos\theta$ and $\sin\theta$. Therefore, the number of crossings of C per unit time for all outward velocities α and β is

$$N_c(\hat{x},\hat{y}) = \int_c \int_0^\infty \int_0^\infty [\alpha\,|\cos\theta| + \beta\,|\sin\theta|] f(\hat{x},\hat{y},\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}s \tag{19}$$

Equation (10) shows that a load component and its velocity are uncorrelated. If the velocities themselves are uncorrelated, then the joint probability density function $f(\hat{x},\hat{y},\alpha,\beta)$ can be expressed as the product of three statistically independent frequency density functions. That is,

$$f(\hat{x},\hat{y},\alpha,\beta) = f(\alpha) f(\beta) f(\hat{x},\hat{y}) \tag{20}$$

and Eq. (19) can be integrated readily with respect to α and β,

$$N_c(\hat{x},\hat{y}) = \int_c \left[\frac{\sigma_\alpha}{\sqrt{2\pi}}\,|\cos\theta| + \frac{\sigma_\beta}{\sqrt{2\pi}}\,|\sin\theta|\right] f(\hat{x},\hat{y})\,\mathrm{d}s \tag{21}$$

Now, it will be noted that $|\cos\theta|\mathrm{d}s$ and $|\sin\theta|\mathrm{d}s$ are the projections of $\mathrm{d}s$ on the y and x axes, respectively. Therefore, the excursions out of the limit design load ellipse, shown in Fig. 2, per unit time can be calculated immediately, since the frequency density is constant on the ellipse. The projections of the ellipse on the y and x axes are $2U_\sigma\bar{A}_y$ and $2U_\sigma\bar{A}_x$, respectively. In integrating around C twice these projections must be used,

$$N_c(\hat{x},\hat{y}) = \left[\frac{\sigma_\alpha}{\sqrt{2\pi}} 4U_\sigma\bar{A}_y + \frac{\sigma_\beta}{\sqrt{2\pi}} 4U_\sigma\bar{A}_x\right] f(\hat{x},\hat{y}) \tag{22}$$

Using Eqs. (1) and (2), where σ_x and σ_y are taken as $\bar{A}_x$ and $\bar{A}_y$, respectively, Eq. (22) becomes

$$\begin{aligned} N_c(\hat{x},\hat{y}) &= \left[\frac{\sigma_\alpha}{\sqrt{2\pi}} 4U_\sigma\bar{A}_y + \frac{\sigma_\beta}{\sqrt{2\pi}} 4U_\sigma\bar{A}_x\right] \times \left[\frac{1}{2\pi\bar{A}_x\bar{A}_y(1-\rho_{xy}^2)^{1/2}}\right] e^{-U_\sigma^2/2} \\ &= \frac{4U_\sigma}{\sqrt{2\pi}(1-\rho_{xy}^2)^{1/2}} \left[\frac{1}{2\pi}\frac{\bar{A}_\alpha}{\bar{A}_x} + \frac{1}{2\pi}\frac{\bar{A}_\beta}{\bar{A}_y}\right] e^{-U_\sigma^2/2} \\ &= \frac{4U_\sigma}{\sqrt{2\pi}(1-\rho_{xy}^2)^{1/2}} [N_{0x} + N_{0y}] e^{-U_\sigma^2/2} \end{aligned} \tag{23}$$

Effect of Velocity Correlations on Boundary Excursions

If the velocities α and β are correlated, then

$$f(\alpha,\beta) = \frac{1}{(2\pi)\sigma_\alpha\sigma_\beta(1-\rho_{\alpha\beta}^2)^{1/2}} \times \exp\left\{\frac{-1}{2(1-\rho_{\alpha\beta}^2)}\left[\frac{\alpha^2}{\sigma_\alpha^2} - \frac{2\rho_{\alpha\beta}\alpha\beta}{\sigma_\alpha\sigma_\beta} + \frac{\beta^2}{\sigma_\beta^2}\right]\right\} \tag{24}$$

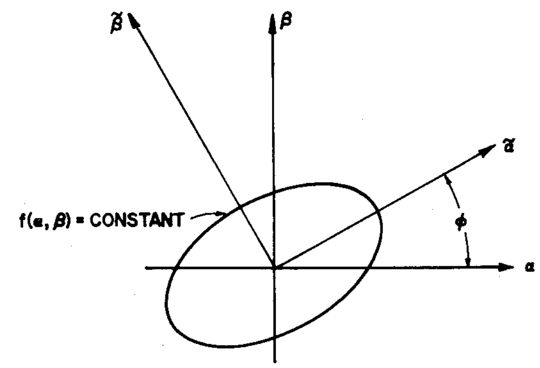

Fig. 4 Transformation of velocity axes.

A coordinate transformation can be used to decouple the velocity components,

$$\alpha = \tilde{\alpha}\cos\phi - \tilde{\beta}\sin\phi \qquad \beta = \tilde{\alpha}\sin\phi + \tilde{\beta}\cos\phi \tag{25}$$

This transformation is indicated in Fig. 4. If the coefficient of the product $\tilde{\alpha}\tilde{\beta}$ is set equal to zero when Eqs. (25) are substituted into Eq. (24), then

$$\tan 2\phi = 2\rho_{\alpha\beta}\sigma_\alpha\sigma_\beta / (\sigma_\alpha^2 - \sigma_\beta^2) \tag{26}$$

and

$$\sigma_{\tilde{\alpha}}^2 = (1-\rho_{\alpha\beta})\left[\frac{\sigma_\alpha^2\sigma_\beta^2}{\sigma_\alpha^2\sin^2\phi - \rho_{\alpha\beta}\sigma_\alpha\sigma_\beta\sin 2\phi + \sigma_\beta^2\cos^2\phi}\right]$$

$$\sigma_{\tilde{\beta}}^2 = (1-\rho_{\alpha\beta})\left[\frac{\sigma_\alpha^2\sigma_\beta^2}{\sigma_\alpha^2\cos^2\phi + \rho_{\alpha\beta}\sigma_\alpha\sigma_\beta\sin 2\phi + \sigma_\beta^2\sin^2\phi}\right] \tag{27}$$

Also

$$\sigma_{\tilde{\alpha}}\sigma_{\tilde{\beta}} = (1-\rho_{\alpha\beta}^2)^{1/2}\sigma_\alpha\sigma_\beta \tag{28}$$

and Eq. (24) is transformed into

$$f(\tilde{\alpha},\tilde{\beta}) = \frac{1}{2\pi\sigma_{\tilde{\alpha}}\sigma_{\tilde{\beta}}}\exp\left[-\frac{1}{2}\left(\frac{\tilde{\alpha}^2}{\sigma_{\tilde{\alpha}}^2} + \frac{\tilde{\beta}^2}{\sigma_{\tilde{\beta}}^2}\right)\right]$$

$$= f(\tilde{\alpha})f(\tilde{\beta}) = \frac{1}{\sqrt{2\pi}\sigma_{\tilde{\alpha}}}\left[\frac{-\tilde{\alpha}^2}{2\sigma_{\tilde{\alpha}}^2}\right]\frac{1}{\sqrt{2\pi}\sigma_{\tilde{\beta}}}\exp\left[\frac{-\tilde{\beta}^2}{2\sigma_{\tilde{\beta}}^2}\right] \tag{29}$$

Equation (19) with correlated α and β can be written in terms of the uncorrelated $\tilde{\alpha}$ and $\tilde{\beta}$ velocities with the angle θ changed to $\theta - \phi$,

$$N_c(\hat{x},\hat{y}) = \int_c \int_0^\infty \int_0^\infty [\tilde{\alpha}\,|\cos(\theta-\phi)|$$

$$+ \tilde{\beta}\,|\sin(\theta-\phi)|\,] f(\hat{x},\hat{y}) f(\tilde{\alpha}) f(\tilde{\beta})\,\mathrm{d}\tilde{\alpha}\,\mathrm{d}\tilde{\beta}\,\mathrm{d}s \tag{30}$$

Equation (21) becomes

$$N_c(\hat{x},\hat{y}) = \int_c \left[\frac{\sigma_{\tilde{\alpha}}}{\sqrt{2\pi}}\,|\cos(\theta-\phi)|\right.$$

$$\left. + \frac{\sigma_{\tilde{\beta}}}{\sqrt{2\pi}}\,|\sin(\theta-\phi)|\,\right] f(\hat{x},\hat{y})\,\mathrm{d}s \tag{31}$$

This expression can be integrated in the same manner as Eq. (22), except that the projections are now on axes rotated by the angle ϕ to the xy coordinates.

Effect of Excitation Level on Limit Design Load Excursion Rate

The number of excursions per unit time out of the limit design load ellipse, Eq. (23), is extremely small, since U_σ is of the order of 75 ft/s (Ref. 2). This is to be expected, because the rms excitation σ_w is unity. If we rewrite Eq. (22) with $U_\sigma = \eta_d \sigma_w$ and $\sigma_\alpha = \bar{A}_\alpha \sigma_w$, etc., then,

$$N_c(\hat{x},\hat{y}) = \frac{4\eta_d}{\sqrt{2\pi}(1-\rho_{xy}^2)^{1/2}}[N_{0x}+N_{0y}]e^{-\eta_d^2/2} \tag{32a}$$

$$= \frac{4U_\sigma}{\sqrt{2\pi}\sigma_w(1-\rho_{xy}^2)^{1/2}}[N_{0x}+N_{0y}]e^{-U_\sigma^2/2\sigma_w^2} \tag{32b}$$

The number of excursions per unit time becomes significant for higher levels of σ_w.

Frequency of Exceedance of a Limit Design Load Ellipsoid

The number of excursions per unit time out of the limit design load ellipsoid for three random load components x, y, and z follows easily from the above development. The area of the ellipse due to the projection of the limit design load ellipsoid on the xy plane is π times the product of the major and minor semiaxes of the ellipse; see Eq. (28). Therefore, the projected area on the xy plane is

$$\text{ellipse area} = \pi(1-\rho_{xy}^2)^{1/2}\sigma_x\sigma_y$$

$$= \pi(1-\rho_{xy}^2)^{1/2}U_\sigma^2\bar{A}_x\bar{A}_y \tag{33}$$

Then the number of excursions per unit time out of the ellipsoid, following the same type of development as Eq. (21), can be calculated, where ds is now an element of surface area on the ellipsoid,

$$N_c(\hat{x},\hat{y},\hat{z}) = \left[\frac{\bar{A}_\gamma}{\sqrt{2\pi}}\sigma_w 2\pi(1-\rho_{xy}^2)^{1/2}U_\sigma^2\bar{A}_x\bar{A}_y\right.$$

$$+ \frac{\bar{A}_\alpha}{\sqrt{2\pi}}\sigma_w 2\pi(1-\rho_{yz}^2)^{1/2}U_\sigma^2\bar{A}_y\bar{A}_z$$

$$\left. + \frac{\bar{A}_\beta}{\sqrt{2\pi}}\sigma_w 2\pi(1-\rho_{xz}^2)^{1/2}U_\sigma^2\bar{A}_x\bar{A}_z\right]$$

$$\times\left[\frac{1}{(2\pi)^{3/2}\bar{A}_x\bar{A}_y\bar{A}_z\sigma_w^3\,|R|^{1/2}}\right]e^{-U_\sigma^2/2\sigma_w^2}$$

$$= \frac{U_\sigma^2}{\sigma_w^2\,|R|^{1/2}}[(1-\rho_{xy}^2)^{1/2}N_{0z} + (1-\rho_{yz}^2)^{1/2}N_{0x}$$

$$+ (1-\rho_{xz}^2)^{1/2}N_{0y}]e^{-U_\sigma^2/2\sigma_w^2} \tag{34}$$

Frequency of Exceedance of a Limit Design Strength Envelope

The number of excursions per unit time out of a limit design strength interaction envelope follows directly from the previous development. In this case, the frequency density $f(\hat{x},\hat{y},\hat{z})$ will vary on the interaction strength surface S, whereas it is constant on the limit design load envelope. Consider an example where the random loading is defined by the parameters in Table 1 and this limit design strength interaction envelope S is defined in the following:

$$(x/F_x)^2 + (y/F_y)^2 + (z/F_z)^2 = 1.0 \tag{35}$$

It is necessary that the limit design load ellipsoid be wholly contained in and tangent to the limit strength envelope for maximum efficiency of the design. Thus, values of U_σ at all points on the strength envelope must not be less than the limit values of U_σ. Obviously, the most efficient design is the design load ellipsoid itself, but all practical structures must be designed for many different design loads. Thus, an efficient design is assured by providing zero margin of safety at one or more critical points on the limit strength surface S.

The points of tangency of the limit design load surface C and the limit design strength surface S are determined by equating the gradients of the two surfaces. The gradient vectors of C and S are

$$\nabla C = \frac{\partial C}{\partial x} i + \frac{\partial C}{\partial y} j + \frac{\partial C}{\partial z} k \tag{36a}$$

$$\nabla S = \frac{\partial S}{\partial x} i + \frac{\partial S}{\partial y} j + \frac{\partial S}{\partial z} k \tag{36b}$$

Thus, the critical points are defined where

$$\nabla C = K \nabla S \tag{37}$$

assuring that the gradient vectors on the two surfaces have the same sense and differ only by a constant, K.

A first estimate of the critical points can be determined by evaluating Eq. (37) at the tangent points on the load ellipsoid with the circumscribing rectangular parallelopiped with faces normal to axes at $x_0 \pm U_\sigma \bar{A}_x$, $y_0 \pm U_\sigma \bar{A}_y$, and $z_0 \pm U_\sigma \bar{A}_z$. These tangent points on planes perpendicular to the x axis are:

$$(x_0 + U_\sigma \bar{A}_x, \quad y_0 + \rho_{xy} U_\sigma \bar{A}_y, \quad z_0 + \rho_{xz} U_\sigma \bar{A}_z)$$

$$(x_0 - U_\sigma \bar{A}_x, \quad y_0 - \rho_{xy} U_\sigma \bar{A}_y, \quad z_0 - \rho_{xz} U_\sigma \bar{A}_z)$$

The limit design load ellipsoid, tangent planes, and tangent points on the planes are indicated in Fig. 5 for the example. The averages of three tangent points in each of the two opposing corners D and F of the parallelopiped are used as first trial values to determine tangency points for the surfaces C and S by Eq. (37).

The surface C is defined by Eq. (4), with U_σ equal to 75 ft/s and σ_i equal to $\bar{A}_i$ given in Table 1. The strength envelope S for the example problem is defined by Eq. (35) for limit design allowable loads defined to provide zero margin of safety at two critical points. The selected allowable loads are

$$F_x = 578, \quad F_y = 722, \quad F_z = 433, \ -149$$

The critical points are (52.7, 506.3, 306.6) and (226.0, −180.0, −132.3). The margins of safety (M.S.) at these points are zero as defined by the following equation:

$$\text{M.S.} = [(x/F_x)^2 + (y/F_y)^2 + (z/F_z)^2]^{-1/2} - 1 \tag{38}$$

The surfaces C and S and the critical points are shown in Fig. 6. Having established C and S, the number of excursions out of the limit strength envelope S per unit time can be deter-

Table 1 Random load parameters, example problem

$\bar{A}_x = 4$	$\bar{A}_\alpha = 150.8$	$\rho_{xy} = -0.36$	$x_0 = 120$
$\bar{A}_y = 5$	$\bar{A}_\beta = 94.3$	$\rho_{xz} = -0.21$	$y_0 = 150$
$\bar{A}_z = 3$	$\bar{A}_\gamma = 18.9$	$\rho_{yz} = 0.84$	$z_0 = 90$
		$U_\sigma = 75$ ft/s	
		$\sigma_w = 20$ ft/s	

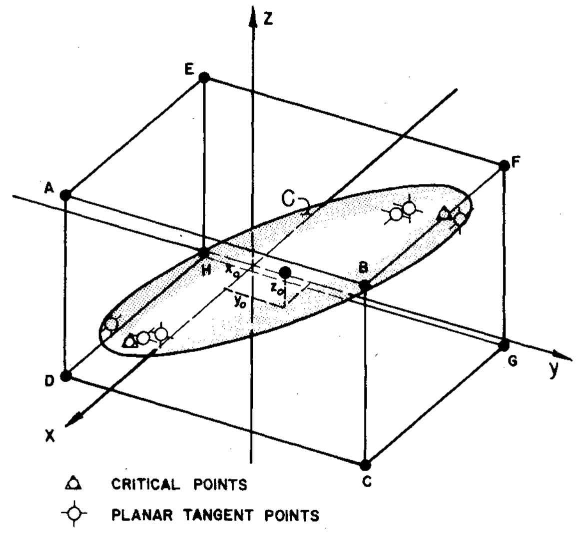

Fig. 5 Limit design load ellipsoid.

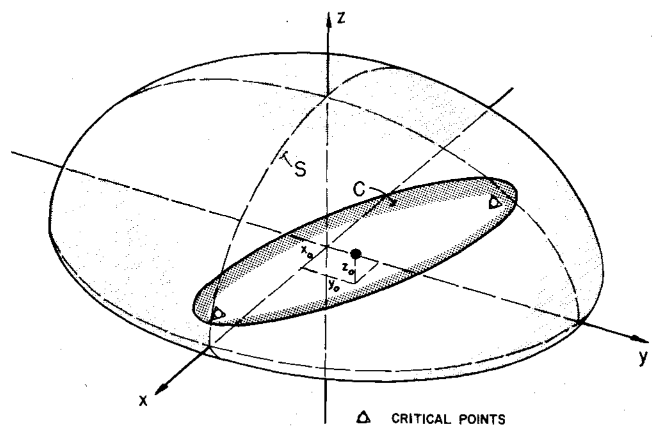

Fig. 6 Surfaces C and S and critical design points, example problem.

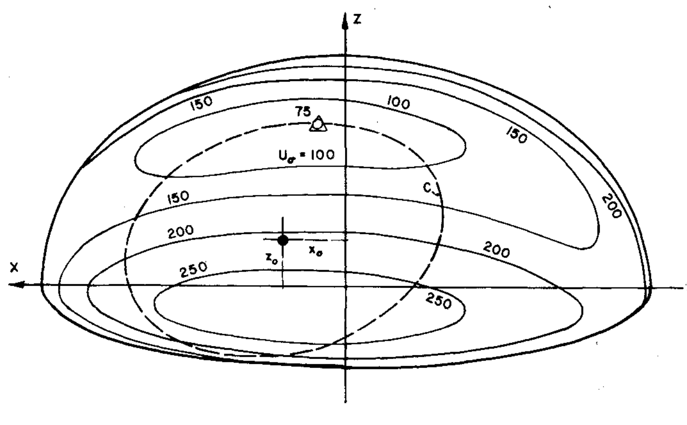

Fig. 7 Map of U_σ on positive y-axis projection of limit strength envelope, example problem.

Table 2 Integral of probability density over projected areas, example problem

$I_{(+x)} = 1.3916 \times 10^{-6}$	$I_{(+y)} = 2.8758 \times 10^{-6}$	$I_{(+z)} = 4.6655 \times 10^{-6}$
$I_{(-x)} = 3.3013 \times 10^{-7}$	$I_{(-y)} = 2.9338 \times 10^{-7}$	$I_{(-z)} = 5.9301 \times 10^{-6}$
Total:		
$I_{(x)} = 1.7217 \times 10^{-6}$	$I_{(y)} = 3.1692 \times 10^{-6}$	$I_{(z)} = 1.0596 \times 10^{-5}$

mined. Figure 7 shows the map of U_σ evaluated on S and projected on a plane perpendicular to the positive y axis. The frequency density is directly related to U_σ. The integral of $f(\hat{x},\hat{y},\hat{z})$ over all six projected surfaces $I_{(+x)}$, $I_{(-x)}$, etc., for σ_w of 20 ft/s provides results shown in Table 2.

The number of excursions per second through the limit strength boundary S shown in Fig. 6 is

$$N_c = \frac{\bar{A}_\alpha \sigma_w}{\sqrt{2\pi}} I_{(x)} + \frac{\bar{A}_\beta \sigma_w}{\sqrt{2\pi}} I_{(y)} + \frac{\bar{A}_\gamma \sigma_w}{\sqrt{2\pi}} I_{(z)}$$

$$= \frac{(150.8)(20)(1.7217)(10)^{-6}}{\sqrt{2\pi}} + \frac{(94.3)(20)(3.1692)(10)^{-6}}{\sqrt{2\pi}}$$

$$+ \frac{(18.9)(20)(1.0596)(10)^{-5}}{\sqrt{2\pi}} = 6.0540(10)^{-3}/\mathrm{s} \qquad (39)$$

Or, the mean time between excursions is the inverse of the result calculated in Eq. (39), that is, 165.18 s when the level of the excitation σ_w is 20 ft/s. The integrals in Table 2 were evaluated numerically over the projected areas using a parabolic interpolation over a network of about 100 points on each projected area such as that illustrated in Fig. 7.

The number of excursions through the limit design load ellipsoid C shown in Figs. 6 and 7 can be calculated directly from Eq. (34) to be 0.17784 excursions per second, or 5.6230 s per excursion.

Concluding Remarks

A procedure was defined to determine the critical design points(s), that is, point(s) where the limit design load ellipsoid C is tangent to the limit design strength interaction envelope S. A critical point will have zero margin of safety. Zero margin of safety will ensure an efficient design; however, the number of excursions of the limit strength boundary should be investigated to determine if the excursion rate is acceptably low.

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